Lecture1. General Properties of the Angular Momentum: eigenvalues and eigenvectors, angular momentum coupling

1 Commutation relations of the angular momentum; angular momentum matrices

The angular momentum **j** of a physical system is a vector, thus in quantum mechanics it is represented by three hermitian operators $\hat{j}_x$, $\hat{j}_y$, $\hat{j}_z$, the projections of the operator of the angular momentum of the system $\hat{j}$ on the coordinate axes x, y and z. These operators satisfy the following commutation relations:

$$[\hat{j}_x, \hat{j}_y] = \mathrm{i}\hat{j}_z \; ; \quad [\hat{j}_y, \hat{j}_z] = \mathrm{i}\hat{j}_x \; ; \quad [\hat{j}_z, \hat{j}_x] = \mathrm{i}\hat{j}_y \; . \tag{1}$$

Here and below, the momentum is given in units of $\hbar$ and $\hat{j}$ is the general notation of the angular momentum and can denote either the orbital angular momentum $\hat{l}$, or the spin $\hat{s}$, or the total angular momentum $\hat{j} = \hat{l} + \hat{s}$ of one particle, or be referred to the same values for a system of particles:

$$\hat{L} = \sum_i \hat{l}_i \; , \hat{S} = \sum_i \hat{s}_i \; , \hat{J} = \sum_i \hat{j}_i \tag{2}$$

Instead of the operators $\hat{j}_x$, $\hat{j}_y$, $\hat{j}_z$, we can introduce three other operators

$$\hat{j}_+ = \hat{j}_x + \mathrm{i}\hat{j}_y \; , \quad \hat{j}_- = \hat{j}_x - \mathrm{i}\hat{j}_y \; , \quad \hat{j}_0 = \hat{j}_z \; , \tag{3}$$

for which the commutation relations take the form:

$$[\hat{j}_0, \hat{j}_\pm] = \pm\hat{j}_\pm \; ; \quad [\hat{j}_+, \hat{j}_-] = 2\hat{j}_0 \; . \tag{4}$$

The operators $\hat{j}_+$ and $\hat{j}_-$ transform in each other under hermitian conjugation:

$$(\hat{j}_+)^\dagger = \hat{j}_- \; ; \quad (\hat{j}_0)^\dagger = \hat{j}_0 \; . \tag{5}$$

From the commutation relations (1) we can get the following results, without the explicit form of the angular momentum operators:

1. There exist the vector states $|j,m\rangle$, which are eigenstates of the both operators $\hat{j}^2$ and $\hat{j}_z$:
$$\begin{aligned} \hat{j}^2|j,m\rangle &= j(j+1)|j,m\rangle \\ \hat{j}_z|j,m\rangle &= m|j,m\rangle \; ; \end{aligned} \tag{6}$$

2. The angular momentum of the quantum mechanical system can take only integer or half-integer values:
$$j = 0 \; ; \frac{1}{2} \; ; 1 \; ; \frac{3}{2} \; ; \ldots \tag{7}$$

3. For a given value of j, the angular momentum projection m can take $(2j+1)$ values:

$$m = j\ ; j-1\ ; j-2\ ; \ldots\ ; -j+2\ ; -j+1\ ; -j\ ; \qquad (8)$$

4. The operators $\hat{j}_+$ and $\hat{j}_-$ being applied to the vectors $|j,m\rangle$ change the values of m and do not change the value of j:

$$\begin{aligned} \hat{j}_-|j,m+1\rangle &= \sqrt{(j-m)(j+m+1)}|j,m\rangle\ , \\ \hat{j}_+|j,m\rangle &= \sqrt{(j-m)(j+m+1)}|j,m+1\rangle \end{aligned} \qquad (9)$$

The relations (1)–(9) are valid not only for the angular momentum operators, but also for any arbitrary hermitian operator, for which the relations (1) hold, for example, for the isospin operator, for various quasispin operators, pseudospin operators, for the operators of U-spin and V-spin, used in the elementary particle physics.

Using the relations (6)–(9) we can easily construct the matrices of the operators $\hat{j}_x$, $\hat{j}_y$ and $\hat{j}_z$ and their combinations. For a given j, these matrices will have the dimensions $(2j+1) \times (2j+1)$.

2 Spherical functions

The eigenfunctions of the square of the orbital angular momentum operator $\hat{l}^2$ and its projection $\hat{l}_z$ on the axis z are the so-called spherical functions $Y_{lm}(\theta,\phi)$ with $l = 0,1,2,\ldots$; $m = l, l-1, \ldots, -l+1, -l$:

$$\begin{aligned} \hat{l}^2 Y_{lm}(\theta,\phi) &= l(l+1)Y_{lm}(\theta,\phi) \\ \hat{l}_z Y_{lm}(\theta,\phi) &= mY_{lm}(\theta,\phi) \end{aligned} \qquad (10)$$

Since in the literature there exist different definitions of the spherical functions, we will use the following expressions:

$$\begin{aligned} Y_{lm}(\theta,\phi) &= \Theta_{lm}(\theta)\,\Phi_m(\phi) \\ \Phi_m(\phi) &= \frac{1}{\sqrt{2\pi}}\exp{(im\phi)} \\ \Theta_{lm}(\theta) &= (-1)^m\sqrt{\frac{(2l+1)(l-m)!}{2(l+m)!}}P^l_m(\cos\theta)\ , \quad m \geq 0\ , \end{aligned} \qquad (11)$$

where $P^l_m(\cos\theta)$ are the associated Legendre polynomials. For $m < 0$, the spherical functions are defined by the equality:

$$Y_{l,-m}(\theta,\phi) = (-1)^m Y^*_{lm}(\theta,\phi)\ . \qquad (12)$$

The following formulae hold for the spherical functions:

1. Under the inversion of the coordinate system ($\hat{P}: \ r \to r, \theta \to \pi - \theta, \phi \to \pi + \phi$) the spherical functions transform as

$$\hat{P} Y_{lm}(\theta, \phi) = (-1)^l Y_{lm}(\theta, \phi) \ ; \tag{13}$$

2. Under the time inversion ($\hat{K}: \ t \to -t$) the spherical functions transform as

$$\hat{K} Y_{lm}(\theta, \phi) = Y^*_{lm}(\theta, \phi) = (-1)^m Y_{l,-m}(\theta, \phi) \ ; \tag{14}$$

3. The spherical functions satisfy the orthogonality relation

$$\int_0^{2\pi} \int_0^{\pi} Y^*_{lm}(\theta, \phi) Y_{l'm'}(\theta, \phi) \sin\theta d\theta d\phi = \delta_{ll'} \delta_{mm'} \ ; \tag{15}$$

4. The spherical functions are a complete set of functions

$$\sum_{lm} Y_{lm}(\theta, \phi) Y^*_{lm}(\theta', \phi') = \delta(\phi - \phi') \delta(\cos\theta - \cos\theta') \ . \tag{16}$$

3 Coupling of two angular momenta: Clebsch-Gordan coefficients

If $\hat{j}_1$ and $\hat{j}_2$ are the angular momenta of two subsystems of a physical system, then the total angular momentum $\hat{J} = \hat{j}_1 + \hat{j}_2$ is quantized and for the given values j_1 and j_2, the values of j are defined by the triangle rule:

$$J = j_1 + j_2 \ , \ j_1 + j_2 - 1 \ , \ \ldots , \ |j_1 - j_2| + 1 \ , \ |j_1 - j_2| \ .$$

Let $\hat{j}_{z1}$ and $\hat{j}_{z2}$ be the operators of the projections on the z-axis of the angular momenta $\hat{j}_1$ and $\hat{j}_2$, respectively. The set of four commuting operators $\{\hat{j}^2_1, \hat{j}_{z1}, \hat{j}^2_2, \hat{j}_{z2}\}$ has the common eigenvectors, expressed by the product state vectors

$$|j_1 m_1 j_2 m_2\rangle \equiv |j_1 m_1\rangle |j_2 m_2\rangle \ . \tag{17}$$

The same quantum system can be equivalently characterized by the other set of four commuting operators $\{\hat{J}^2, \hat{J}_z, \hat{j}^2_1, \hat{j}^2_2\}$. The eigenvectors common for the latter set of operators can be expressed in terms of the eigenvectors (17) as

$$|j_1 j_2; JM\rangle = \sum_{m_1, m_2} (j_1 m_1 j_2 m_2 | JM) |j_1 m_1 j_2 m_2\rangle \ . \tag{18}$$

The transformation coefficients $(j_1 m_1 j_2 m_2 | JM)$ are called the Clebsch-Gordan coefficients.

The inverse relation also holds:

$$|j_1 m_1 j_2 m_2\rangle = \sum_{JM} (j_1 m_1 j_2 m_2 | JM) |j_1 j_2; JM\rangle \ . \tag{19}$$

The basic properties of the Clebsch-Gordan coefficients:

1. The Clebsch-Gordan coefficients are non-zero only if

$$
\begin{array}{l}
|j_1 - j_2| \le J \le j_1 + j_2 \ ; \\
|J - j_2| \le j_1 \le J + j_2 \ ; \\
|J - j_1| \le j_2 \le J + j_1 \ ; \ .
\end{array} \tag{20}
$$

   and $M = m_1 + m_2$.

2. The following orthogonality relations hold

$$
\begin{array}{l}
\sum_{m_1,m_2} (j_1 m_1 j_2 m_2|JM)(j_1 m_1 j_2 m_2|J'M') = \delta_{J,J'}\delta_{M,M'} \\
\sum_{JM} (j_1 m_1 j_2 m_2|JM)(j_1 m_1' j_2 m_2'|JM) = \delta_{m_1,m_1'}\delta_{m_2,m_2'}
\end{array} \tag{21}
$$

3. The following symmetry properties under permutation of the angular momentum operators hold

$$
\begin{array}{rcl}
(j_1 m_1 j_2 m_2|JM) & = & (-1)^{j_1+j_2-J}(j_2 m_2 j_1 m_1|JM) \\
 & = & (-1)^{j_2+m_2}\sqrt{\frac{2J+1}{2j_1+1}}(J-M j_2 m_2|j_1 - m_1) \\
 & = & (-1)^{j_1-m_1}\sqrt{\frac{2J+1}{2j_2+1}}(j_1 m_1 J - M|j_2 - m_2) \\
 & = & (-1)^{j_1+j_2-J}(j_1 - m_1 j_2 - m_2|J - M) \ ;
\end{array} \tag{22}
$$

4. If $J = M = 0$, then

$$
(j_1 m_1 j_2 m_2|00) = \delta_{j_1,j_2}\delta_{m_1,-m_2}\frac{(-1)^{j_1-m_1}}{\sqrt{2j_1+1}} \ ; \tag{23}
$$

The Clebsch-Gordan coefficients are related to the $3j$-symbols as

$$
(j_1 m_1 j_2 m_2|j_3 - m_3) = (-1)^{j_1-j_2-m_3}\sqrt{2j_3+1}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{24}
$$

The tables of the Clebsch-Gordan coefficients can be found in [3, 6, 7]. There exist Fortran programs and in-built functions in the Mathematica package.

4 Coupling of three angular momenta: $6j$-symbols

In the case of a system described by three independent angular momentum operators $\hat{j}_1$, $\hat{j}_2$ and $\hat{j}_3$, one can again form the total angular momentum operator $\hat{J} = \hat{j}_1 + \hat{j}_2 + \hat{j}_3$. The six commuting operators $\{\hat{j}_1^2, \hat{j}_{z1}, \hat{j}_2^2, \hat{j}_{z2}, \hat{j}_3^2, \hat{j}_{z3}\}$ have a common set of the eigenvectors, expressed by the product state vectors

$$
|j_1 m_1\rangle|j_2 m_2\rangle|j_3 m_3\rangle \ . \tag{25}
$$

The same quantum system can be equivalently characterized by any of the following sets of six commuting operators:

$$
\begin{array}{l}
\{\hat{J}^2, \hat{J}_z, \hat{j}_1^2, \hat{j}_2^2, \hat{j}_3^2, \hat{J}_{12}^2\} \\
\{\hat{J}^2, \hat{J}_z, \hat{j}_1^2, \hat{j}_2^2, \hat{j}_3^2, \hat{J}_{23}^2\} \\
\{\hat{J}^2, \hat{J}_z, \hat{j}_1^2, \hat{j}_2^2, \hat{j}_3^2, \hat{J}_{13}^2\}
\end{array} \tag{26}
$$

with the sets of common eigenvectors

$$\begin{array}{l} |(j_1 j_2)J_{12} j_3; JM\rangle \\ |j_1(j_2 j_3)J_{23}; JM\rangle \\ |(j_1 j_3)J_{13} j_2; JM\rangle \ , \end{array} \tag{27}$$

respectively.

Note, that the ordering of the angular momenta is important:

$$\begin{aligned} |(j_1 j_2)J_{12} j_3; JM\rangle &= (-1)^{j_1+j_2-J_{12}}|(j_2 j_1)J_{12} j_3; JM\rangle \\ &= (-1)^{J_{12}+j_3-J}|j_3(j_1 j_2)J_{12}; JM\rangle \end{aligned} \tag{28}$$

(these property follows simply from the properties of the Clebsch-Gordan coefficients under permutation of the angular momenta).

The transformations between one set of eigenvectors to another one can be written formally as

$$|j_1(j_2 j_3)J_{23}; JM\rangle = \sum_{J_{12}} U(j_1 j_2 J j_3; J_{12} J_{23})|(j_1 j_2)J_{12} j_3; JM\rangle \tag{29}$$

The coefficients in this expression are called Racah coefficients and they are related to the $6j$-symbols as

$$U(j_1 j_2 J j_3; J_{12} J_{23}) = (-1)^{j_1+j_2+j_3+J}\sqrt{(2J_{12}+1)(2J_{23}+1)} \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & J & J_{23} \end{Bmatrix} \tag{30}$$

The special case of the $6j$-symbols is useful:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ 0 & l_2 & l_3 \end{Bmatrix} = \delta_{l_2,j_3}\delta_{l_3,j_2}\frac{(-1)^{j_1+j_2+j_3}}{\sqrt{(2j_2+1)(2j_3+1)}} \ . \tag{31}$$

5 Coupling of four angular momenta: $9j$-symbols

Similarly, we can construct the total angular momentum operator corresponding to the sum of four independent angular momentum operators as $\hat{J} = \hat{j}_1 + \hat{j}_2 + \hat{j}_3 + \hat{j}_4$. The eight commuting operators $\{\hat{j}_1^2, \hat{j}_{z1}, \hat{j}_2^2, \hat{j}_{z2}, \hat{j}_3^2, \hat{j}_{z3}, \hat{j}_4^2, \hat{j}_{z4}\}$ have a common set of the eigenvectors, expressed by the product state vectors

$$|j_1 m_1\rangle|j_2 m_2\rangle|j_3 m_3\rangle|j_4 m_4\rangle \ . \tag{32}$$

The same quantum system can be equivalently characterized by any of the following sets of four commuting operators with only pairwise coupling of the angular momenta:

$$\begin{array}{l} \{\hat{J}^2, \hat{J}_z, \hat{j}_1^2, \hat{j}_2^2, \hat{j}_3^2, \hat{j}_4^2, \hat{J}_{12}^2, \hat{J}_{34}^2\} \\ \{\hat{J}^2, \hat{J}_z, \hat{j}_1^2, \hat{j}_2^2, \hat{j}_3^2, \hat{j}_4^2, \hat{J}_{13}^2, \hat{J}_{24}^2\} \\ \{\hat{J}^2, \hat{J}_z, \hat{j}_1^2, \hat{j}_2^2, \hat{j}_3^2, \hat{j}_4^2, \hat{J}_{14}^2, \hat{J}_{23}^2\} \end{array} \tag{33}$$

with the sets of common eigenvectors

$$
\begin{array}{l}
|(j_1j_2)J_{12}(j_3j_4)J_{34};JM\rangle \\
|(j_1j_3)J_{13}(j_2j_4)J_{24};JM\rangle \\
|(j_1j_4)J_{14}(j_2j_3)J_{23};JM\rangle\ ,
\end{array}
\tag{34}
$$

respectively.

The transformation from one set of eigenvectors to another one can be written formally as

$$
|(j_1j_3)J_{13}(j_2j_4)J_{24};JM\rangle = \sum_{J_{12}J_{34}} \sqrt{(2J_{12}+1)(2J_{34}+1)(2J_{13}+1)(2J_{24}+1)} \left\{ \begin{array}{ccc} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{array} \right\} \times |(j_1j_2)J_{12}(j_3j_4)J_{34};JM\rangle
\tag{35}
$$

The special case of the $9j$-symbol is useful:

$$
\left\{ \begin{array}{ccc} j_1 & j_2 & j \\ j_3 & j_4 & j \\ k & k & 0 \end{array} \right\} = (-1)^{j_2+j_3+j+k} \sqrt{(2j+1)(2k+1)} \left\{ \begin{array}{ccc} j_1 & j_2 & j \\ j_4 & j_3 & k \end{array} \right\}
\tag{36}
$$

We can use the general expression (35) when for example recoupling from a jj-scheme basis to a LS-scheme basis in the case of two $s=1/2$ fermions:

$$
|(l_1l_2)L(\tfrac{1}{2}\tfrac{1}{2})S;JM\rangle = \sum_{j_1j_2} \sqrt{(2L+1)(2S+1)(2j_1+1)(2j_2+1)} \left\{ \begin{array}{ccc} l_1 & l_2 & L \\ \frac{1}{2} & \frac{1}{2} & S \\ j_1 & j_2 & J \end{array} \right\} \times |(l_1\tfrac{1}{2})j_1(l_2\tfrac{1}{2})j_2;JM\rangle
\tag{37}
$$

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