Lecture 11. Continuous Groups

1 Continuous matrix groups

In this section we shall consider the continuous groups whose elements can be labelled by a finite set of continuously varying real parameters. The typical element of an n-parameter group looks like $G(x_1, x_2, \ldots, x_n) \equiv G(x)$. The multiplication of two elements can be written as

$$G(x_1, x_2, \ldots, x_n) \cdot G(y_1, y_2, \ldots, y_n) = G(z_1, z_2, \ldots, z_n) \tag{1}$$

where n parameters $z_1, z_2, \ldots, z_n$ are functions of the parameters $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$. The multiplication table for continuous groups transforms in a set of n real functions depending on $2n$ real parameters:

$$\begin{array}{l} z_1 = f_1(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)\ , \\ z_2 = f_2(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)\ , \\ \ldots \\ z_n = f_n(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)\ . \end{array} \tag{2}$$

If the functions $f_i(x, y)$ are continuous and analytical (there exists derivatives of all orders), then the group is called a *Lie group*.

Let us consider the transformations in the vicinity of the identity element of the continuous group, $E \equiv G(0, 0, \ldots, 0)$. For all x, $G(0) \cdot G(x) = G(x)$.

The n operators

$$\begin{array}{l} \hat{X}_1 = \lim\limits_{\epsilon \to 0} \dfrac{G(\epsilon, 0, 0, 0 \ldots, 0) - G(0, 0, 0, 0 \ldots, 0)}{\epsilon}\ , \\ \hat{X}_2 = \lim\limits_{\epsilon \to 0} \dfrac{G(0, \epsilon, 0, 0, \ldots, 0) - G(0, 0, 0, 0 \ldots, 0)}{\epsilon}\ , \\ \ldots \\ \hat{X}_n = \lim\limits_{\epsilon \to 0} \dfrac{G(0, 0, \ldots, 0, \epsilon) - G(0, 0, 0, \ldots, 0)}{\epsilon}\ . \end{array} \tag{3}$$

are called *infinitesimal operators* of the group.

Each transformation of a parameter x_i on a small value ϵ can be expressed as

$$G(0, 0, \ldots, \epsilon, \ldots, 0) \approx G(0, 0, 0, 0 \ldots, 0) + \epsilon \hat{X}_i\ . \tag{4}$$

Any finite transformation a can be written as successive infinitesimal transformations

$$G(0, 0, \ldots, a, \ldots, 0) = \left(G(0, 0, 0, 0 \ldots, 0) + \frac{a}{N} \hat{X}_i \right)^N \to \exp\left(a \hat{X}_i\right) \quad \text{for} \quad N \to \infty\ . \tag{5}$$

This result can be generalized for any arbitrary transformation as

$$G(a_1, a_2, \ldots, a_n) = exp(\sum_i a_i \hat{X}_i)\ . \tag{6}$$

Thus, in order to generate all elements of a group Lie, it is sufficient to know all infinitesimal operators.

Properties of infinitesimal operators

1. The commutator of two infinitesimal operators of a continuous group is again a superposition of infinitesimal operators

$$\left[\hat{X}_i, \hat{X}_j\right] = \sum_k c_{ij}^k \hat{X}_k \ , \tag{7}$$

where the coefficients c_{ij}^k are called the *structure constants.*

2. The infinitesimal operators of a Lie group satisfy the Jacobi identity:

$$\left[\left[\hat{X}_i, \hat{X}_j\right], \hat{X}_k\right] + + \left[\left[\hat{X}_j, \hat{X}_k\right], \hat{X}_i\right] + \left[\left[\hat{X}_k, \hat{X}_i\right], \hat{X}_j\right] = 0 \ . \tag{8}$$

All possible linear combinations of the type

$$c_1\hat{X}_1 + c_2\hat{X}_2 + \ldots + c_n\hat{X}_n \tag{9}$$

form the so-called *Lie algebra* g associated with the Lie group **G**.

The set of functions $f_i^{(\alpha)}(r)$ is said to transform according to the irreducible representation $D^{(\alpha)}(G)$ if

$$f_i^{(\alpha)\prime}(r) = \sum_{j=1}^{l_\alpha} D_{ji}^{(\alpha)}(G) f_j^{(\alpha)}(r) \tag{10}$$

where $D_{ji}^{(\alpha)}(G)$ are the matrix elements of a representation $D^{(\alpha)}(G)$ of the group **G**.

For the continuous group it is sufficient to show that the following relation holds:

$$\hat{X}_q f_i^{(\alpha)}(r) = \sum_{j=1}^{l_\alpha} (X_q)_{ji}^{(\alpha)} f_j^{(\alpha)}(r) \ , \tag{11}$$

where $(X_q)_{ji}^{(\alpha)}$ are the matrix elements of the operator $\hat{X}_q$ in the representation $D^{(\alpha)}(G)$.

Similarly, an irreducible set of operators $\hat{T}_i^{(\alpha)}$ transforms as

$$\hat{T}_i^{(\alpha)} \xrightarrow{G} \hat{T}\,'^{(\alpha)}_i = \hat{D}(G)\hat{T}_i^{(\alpha)}\hat{D}^{-1}(G) = \sum_j D_{ji}^{(\alpha)}(G)\hat{T}_j^{(\alpha)} \ , \tag{12}$$

where $D_{ji}^{(\alpha)}(G)$ are the matrix elements of the irreducible representation $D^{(\alpha)}(G)$. For the continuous groups the condition (12) is equivalent to

$$\left[\hat{X}_q, \hat{T}_i^{(\alpha)}\right] = \sum_{j=1}^{l_\alpha} (X_q)_{ji}^{(\alpha)} \hat{T}_j^{(\alpha)} \ , \tag{13}$$

This means that if the Hamiltonian is invariant with respect to a continuous group, it should commute with all group infinitesimal operators.

The operators $\hat{C}$ which commute with all group infinitesimal operators,

$$\left[\hat{C}, \hat{X}_q\right] = 0 \quad \text{for all } q \ , \tag{14}$$

are called *Casimir operators*, or *Casimir invariants.*

2 SO(2)

2.1 Infinitesimal operators

The rotation group in two dimensions **SO(2)** is characterized by one parameter, the angle of rotation a $(0 \leq a < 2\pi)$:

$$D(a) = \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix} . \tag{15}$$

To get the expression for the generator in a *differential form*, we will follow the definition (3) and consider the rotation on a small angle around z-axis

$$\begin{array}{l} \vec{e}_x\,' \approx \vec{e}_x + a\vec{e}_y \; , \\ \vec{e}_y\,' \approx -a\vec{e}_x + \vec{e}_y \end{array} \tag{16}$$

From (3) we have the expression in the cartesian coordinates

$$\hat{X} f(x,y) = \lim_{a\to 0} \frac{1}{a} \left\{ f(x+ay, -ax+y) - f(x,y) \right\} = \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \right) f(x,y) \; , \tag{17}$$

i.e. the generator is

$$\hat{X} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = -\frac{\partial}{\partial \phi} \; , \tag{18}$$

where ϕ is a polar angle. Since $\frac{\partial}{\partial \phi} = \mathrm{i}\hat{J}_z$

$$\hat{X} = -\mathrm{i}\hat{J}_z \; . \tag{19}$$

So, $\hat{J}_z$-component of the angular momentum operator is an SO(2) generator.

From (5) follows, that the rotation on the finite angle is

$$\hat{D}(a) = \exp\left(-a\frac{\partial}{\partial \phi}\right) \; , \tag{20}$$

that agrees with what was obtained in lecture 4.

To get the form of an infinitesimal operator in a matrix form, e.g in the basis $(\vec{e}_x, \vec{e}_y)$, we will again consider the rotations on a small angle $a \approx 0$:

$$D(a) \approx \begin{pmatrix} 1 & -a \\ a & 1 \end{pmatrix} = 1 + \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} = 1 + a \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{21}$$

The operator of infinitesimal rotations in a *matrix form* is

$$\hat{X} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{22}$$

2.2 Irreducible representations

Any function of the type

$$\psi^{(m)}(\phi) = A_m \exp(im\phi) \tag{23}$$

where m is a number, can serve as a basis of the 1-dimensional irreducible representations which will be of the form

$$D^m(a) = \exp(-ima)\,. \tag{24}$$

The group **SO(2)** has an infinite number of the 1-dimensional irreducible representations (24). If m is real and integer, then these representations are unitary and continuous for $0 \leq a < 2\pi$. If m is real and half-integer, then these representations are unitary and continuous for $0 \leq a < 4\pi$.

The character of the elements in each representation $D^m(a)$ is thus

$$\chi^{(m)}(a) = \exp(-ima)\,. \tag{25}$$

3 SO(3)

3.1 Infinitesimal operators

The rotation group in three dimensions **SO(3)** is a three parameter group.

Let us first find the generators in a differential form. Let us consider rotations around each of three axes through the small angle a. First, around x-axis:

$$\begin{aligned}\vec{e}_x{}' &= \vec{e}_x\,,\\ \vec{e}_y{}' &= \cos a\vec{e}_y + \sin a\vec{e}_z \approx \vec{e}_y + a\vec{e}_z\\ \vec{e}_z{}' &= -\sin a\vec{e}_y + \cos a\vec{e}_z \approx -a\vec{e}_y + \vec{e}_z\,.\end{aligned} \tag{26}$$

From (3) we have

$$\hat{X}_x f(x,y) = \lim_{a\to 0}\frac{1}{a}\left\{f(x, y+az, -ay+z) - f(x,y,z)\right\} = \left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) f(x,y,z)\,. \tag{27}$$

Thus, we get

$$\hat{X}_x = \left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)\,. \tag{28}$$

Similarly, considering the rotations around y and z axes, we can get the two other infinitesimal operators

$$\hat{X}_y = \left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right)\,, \quad \hat{X}_z = \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\,. \tag{29}$$

We can now calculate the commutation relations between infinitesimal operators (28)–(29), thus getting

$$[\hat{X}_x, \hat{X}_y] = \hat{X}_z\,; \quad [\hat{X}_y, \hat{X}_z] = \hat{X}_x\,; \quad [\hat{X}_z, \hat{X}_x] = \hat{X}_y\,. \tag{30}$$

These relations give us the values of the structure constants for this group.

It is interesting to note, that if we introduce operators $\hat{J}_q = \mathrm{i}\hat{X}_q$, then the commutation relations (30) transform to

$$[\hat{J}_x, \hat{J}_y] = \mathrm{i}\hat{J}_z\ ; \quad [\hat{J}_y, \hat{J}_z] = \mathrm{i}\hat{J}_x\ ; \quad [\hat{J}_z, \hat{J}_x] = \mathrm{i}\hat{J}_y\ , \tag{31}$$

i.e. the components of the angular momentum operator can be considered as **SO(3)** infinitesimal operators.

In the differential form they look like

$$\hat{J}_x = \mathrm{i}\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)\ , \quad \hat{J}_y = \mathrm{i}\left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right)\ , \quad \hat{J}_z = \mathrm{i}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\ . \tag{32}$$

To get the matrix form of the generators in the basis $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$, let us again start with the rotations on a small angle $a \approx 0$ around each of the axes (x, y, z). For x-axis we have from (26) that the operator of rotation is

$$D_x(a) \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -a \\ 0 & a & 1 \end{pmatrix} = 1 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & a & 0 \end{pmatrix} = 1 + a\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{33}$$

from which we find an operator of infinitesimal rotations around x-axis in a *matrix form* in a cartesian basis:

$$\hat{X}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\ . \tag{34}$$

Similarly, we can get

$$\hat{X}_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\ , \quad \hat{X}_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ . \tag{35}$$

The matrix form of the operators $\hat{J}_q$ can be easily constructed in a spherical basis, or using the functions $|jm\rangle$. For example, for $j = 1/2$ we have

$$\hat{J}_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \quad \hat{J}_y = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\ , \quad \hat{J}_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ . \tag{36}$$

Note that for $j = 1/2$, the operators $\hat{J}_q = \frac{1}{2}\hat{\sigma}_q$, where $\hat{\sigma}_q$ are the Pauli matrices.

For $j = 1$, the infinitesimal operators in a matrix form are

$$\hat{J}_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\ , \quad \hat{J}_y = \frac{\mathrm{i}}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\ , \quad \hat{J}_z = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\ . \tag{37}$$

3.2 Irreducible representations

The group **SO(3)** is a three-parameter group (three parameters are necessary in order to characterize a rotation, e.g. three Euler angles). Any rotation in a 3-dimensional space through an arbitrary angle around an arbitrary oriented axis can be represented by an operator

$$\hat{D}(\alpha,\beta,\gamma)=\exp(-\mathrm{i}\alpha\hat{J}_z)\exp(-\mathrm{i}\beta\hat{J}_y)\exp(-\mathrm{i}\gamma\hat{J}_z)\ , \tag{38}$$

where $0\le\alpha\le 2\pi$, $0\le\beta<\pi$, $0\le\gamma<2\pi$. A very well-known matrix representation of these operators can be obtained in the basis spanned by the eigenfunctions of the angular momentum operator $\hat{J}^2$ and its projection $\hat{J}_z$,

$$\begin{aligned}\hat{J}^2|jm\rangle&=j(j+1)|jm\rangle\\ \hat{J}_z|jm\rangle&=m|jm\rangle\ ,\end{aligned} \tag{39}$$

and this is the Wigner D-functions

$$D^{(j)}_{mm'}(\alpha,\beta,\gamma)\equiv\langle jm|\hat{D}(\alpha,\beta,\gamma)|jm'\rangle=\langle jm|\exp(-\mathrm{i}\alpha\hat{J}_z)\exp(-\mathrm{i}\beta\hat{J}_y)\exp(-\mathrm{i}\gamma\hat{J}_z)|jm'\rangle\ . \tag{40}$$

Here j can be either integer or half-integer, $j=0,1/2,2,3/2,\ldots$, and for a given j, the dimension of the representation $D^{(j)}$ is equal to $(2j+1)$ because $m,m'=-j,-j+1,\ldots,j-1,j$.

Since the functions $|jm\rangle$ are eigenfunctions of $\hat{J}_z$, the expression (40) can be further simplified:

$$\begin{aligned}D^{(j)}_{mm'}(\alpha,\beta,\gamma)&=\langle jm|\exp(-\mathrm{i}\alpha\hat{J}_z)\exp(-\mathrm{i}\beta\hat{J}_y)\exp(-\mathrm{i}\gamma\hat{J}_z)|jm'\rangle=\\ \exp(-\mathrm{i}\alpha m)\langle jm|\exp(-\mathrm{i}\beta\hat{J}_y)|jm'\rangle\exp(-\mathrm{i}\gamma m')&=\exp(-\mathrm{i}\alpha m)d^{(j)}_{mm'}(\beta)\exp(-\mathrm{i}\gamma m')\ .\end{aligned} \tag{41}$$

As an example, let us calculate $d^{(j)}_{mm'}(\beta)$ for $j=1/2$.

$$\exp(-\mathrm{i}\beta\hat{J}_y)=\exp(-\mathrm{i}\frac{\beta}{2}\hat{\sigma}_y)=\cos\frac{\beta}{2}\hat{E}-\mathrm{i}\sin\frac{\beta}{2}\hat{\sigma}_y\ , \tag{42}$$

where $\hat{\sigma}_y$ is the Pauli matrix. Thus, we get

$$d^{(1/2)}_{mm'}(\beta)=\begin{pmatrix}\cos\frac{\beta}{2} & \sin\frac{\beta}{2}\\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\ . \tag{43}$$

Physical meaning of the D-functions. Let us act with a rotation operator on the vector state $|jm\rangle$:

$$D^{(j)}(\alpha,\beta,\gamma)|jm\rangle=\sum_{mm'}D^{(j)}_{m'}(\alpha,\beta,\gamma)|jm'\rangle \tag{44}$$

So, if the system characterized by an angular momentum j had a projection m on z-axis, then the probability that in this state the projection of the angular momentum of the system on the axis which is turned on the angle (α,β,γ) will be equal to m' is given by $|D^{(j)}_{mm'}(\alpha,\beta,\gamma)|^2$.

Let us find the characters of $\mathbf{SO(3)}$ elements in the irreducible representation $D^{(j)}$. Since all rotations on the same angle around differently oriented axes belong to the same class, i.e. they have the same character, then let is choose the rotation axis to coincide with z-axis. The matrix of a rotation operator will be then:

$$D^{(j)}(\alpha,0,0)=\begin{pmatrix}\exp(-\mathrm{i}j\alpha) & 0 & 0 & \ldots & 0\\ 0 & \exp(-\mathrm{i}(j-1)\alpha) & 0 & \ldots & 0\\ 0 & \ldots & \ldots & \ldots & 0\\ 0 & \ldots & 0 & 0 & \exp(\mathrm{i}j\alpha)\end{pmatrix}\ . \tag{45}$$

The trace of this matrix is

$$\chi^{(j)}(\alpha,0,0)=\sum_{m=-j}^{j}\exp(-\mathrm{i}m\alpha)=\frac{\sin(j+\frac{1}{2})\alpha}{\sin\frac{1}{2}\alpha}\ . \tag{46}$$

3.3 Casimir invariant

Casimir operator in the group **SO(3)** is the sum of the squares of all infinitesimal operators, i.e.

$$\hat{J}^2=\hat{J}_x^2+\hat{J}_y^2+\hat{J}_z^2\ , \tag{47}$$

because it commutes with all infinitesimal operators:

$$[\hat{J}^2,\hat{J}_x]=[\hat{J}^2,\hat{J}_y]=[\hat{J}^2,\hat{J}_z]=0\ . \tag{48}$$

4 SU(2)

4.1 Infinitesimal operators

The group **SU(2)** is a three parameter group of unitary (2×2) matrices whose determinant equals to 1. It can be parametrized as

$$U=\begin{pmatrix}\exp(\mathrm{i}\xi)\cos\eta & \exp(\mathrm{i}\zeta)\sin\eta\\ -\exp(-\mathrm{i}\zeta)\sin\eta & \exp(-\mathrm{i}\xi)\cos\eta\end{pmatrix}\ . \tag{49}$$

Let us denote the vectors of the 2-dimensional space as

$$\begin{pmatrix}x_1\\ x_2\end{pmatrix} \tag{50}$$

Similarly to **SO(3)**, we can get the infinitesimal operators in a differential form

$$\hat{X}_\xi=\mathrm{i}\left(x_1\frac{\partial}{\partial x_1}-x_2\frac{\partial}{\partial x_2}\right)\ ,\quad \hat{X}_\eta=x_2\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial x_2}\ ,\quad \hat{X}_\zeta=\mathrm{i}\left(x_2\frac{\partial}{\partial x_1}+x_1\frac{\partial}{\partial x_2}\right)\ . \tag{51}$$

To get the matrix form of the infinitesimal operators of a matrix group, we can apply the definition (3) to matrices themselves, since the matrix differentiation is a well-defined operation. Thus we get:

$$\hat{X}_\xi=\begin{pmatrix}\mathrm{i} & 0\\ 0 & -\mathrm{i}\end{pmatrix}\ ,\quad \hat{X}_\eta=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\ ,\quad \hat{X}_\zeta=\begin{pmatrix}0 & \mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}\ , \tag{52}$$

Note, that matrices $\sigma_q=\mathrm{i}X_q$ coincide with the Pauli matrices.

It is clear that the commutation relations between infinitesimal operators (51), or (52),

$$[\hat{X}_\xi,\hat{X}_\eta]=\hat{X}_\zeta\ ,\quad [\hat{X}_\eta,\hat{X}_\zeta]=\hat{X}_\xi\ ,\quad [\hat{X}_\zeta,\hat{X}_\xi]=\hat{X}_\eta \tag{53}$$

coincide with the commutation relation of the **SO(3)** infinitesimal operators (30). In particular, this means that three components of the angular momentum operator $\hat{J}_q$ are also **SU(2)** infinitesimal operators.

The commutation relations between infinitesimal operators of a group define the corresponding algebras. Equivalence of the commutation relations of the groups **SO(3)** and **SU(2)** means that the algebras $so(3)$ and $su(2)$ are isomorphic:

$$so(3) \simeq su(2) \tag{54}$$

However, it is necessary to remember that the groups **SO(3)** and **SU(2)** are different. To each element of **SU(2)**, we can put into correspondence an element of **SO(3)**. However, the group **SU(2)** is a larger group. Namely, the elements $D(0,0,0)$ and $D(0,2\pi,0)$ are the same within **SO(3)**, but they correspond to two different elements of **SU(2)**:

$$D(0,0,0) \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad D(0,2\pi,0) \to \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{55}$$

4.2 Irreducible representations

From isomorphism between $so(3)$ and $su(2)$ algebras, it follows that the irreducible representations of the group **SU(2)** coincide with those of the group **SO(3)**, i.e. we can again use D-functions $D^{(j)}_{mm'}(\alpha,\beta,\gamma)$.

5 Isospin and the group SU(2)

Let us consider a 2-dimensional vector space spanned by two vector states of a nucleon, proton and neutron:

$$|p\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad |n\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{56}$$

Let us consider the group **U(2)** of unitary transformations of this space (non-unitary transformations would not preserve the normalization of these states). Since the transformations which change simultaneously the phase of both proton and neutron are of no physical interest, we will restrict ourselves only to the unitary transformations with determinant 1, i.e. to the group **SU(2)**. Its infinitesimal operators can be represented by three matrices:

$$\tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \tau_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} , \quad \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{57}$$

(note that within **U(2)** we should have also added the (2×2) identity matrix as the fourth infinitesimal operator of the group). The irreducible representations of this group can be denoted by $D^{(T)}$, where $T = 0, 1/2, 1, 3/2, \ldots$ is called an *isospin*.

The operators $t_x = \frac{1}{2}\tau_x$, $t_y = \frac{1}{2}\tau_y$ and $t_z = \frac{1}{2}\tau_z$ represent the components of the isospin operator with $t = 1/2$. For example, proton $|p\rangle$ and neutron $|n\rangle$ are the eigenstates of t^2 and t_z with $t = 1/2$ and $m_t = \pm 1/2$, respectively.

Let us consider the system of A nucleons. The vector space of all possible charge states of such a system has 2^A dimensions. The simultaneous transformations of this vector space can be represented as

$$U = \prod_{i=1}^{A} U(i) \tag{58}$$

where $U(i)$ is the element of the group $\mathbf{SU(2)}$ acting in the space of vector states of ith nucleon. The infinitesimal operators which generate U are

$$\hat{T} = \sum_{i=1}^{A} \hat{t}(i) \tag{59}$$

From here it follows that the operators $\hat{T}$ satisfy the same commutation relations as (2×2) matrices $\hat{t}(i)$, which means that we deal with a 2^A-dimensional representation of the group $\mathbf{SU(2)}$. This representation is reducible and it can be decomposed into irreducible components $D^{(T)}$, which leads to the classification of the nuclear eigenstates according to an isospin quantum number T. Each of these representations is $(2T+1)$-fold degenerate. In a system of A nucleons the total isospin can reach maximum value $T = \frac{1}{2}A$.

For a system consisting of Z protons and N neutrons, the $M_T = \frac{1}{2}(Z-N)$ and thus conserves. So, different components of the isospin multiplet with a given T correspond to different nuclei with the same A, but different values $(Z-N)$.

6 General comments on unitary and orthogonal groups

6.1 U(n)

6.1.1 Infinitesimal operators

The group $\mathbf{U(n)}$ is an n^2 parameter group of unitary transformations in an n-dimensional complex space $(x_1, x_2, \ldots, x_n)$:

$$x_i' = \sum_{j=1}^{n} U_{ij} x_j \ , \quad i = 1, 2, \ldots, n, \tag{60}$$

where U is a unitary matrix: $UU^\dagger = E$.

An infinitesimal unitary transformation can be written as

$$U \approx 1 + \mathrm{i}\epsilon S \ , \tag{61}$$

where S is a hermitian matrix. The matrix S represents an operator which acting on an arbitrary function $f(x_i)$ gives

$$Sf(x_i) = f(x_i') = f(1 + \mathrm{i}\epsilon \sum_{j=1}^{n} S_{ij} x_j) = f(x_i) + \mathrm{i}\epsilon \sum_{i,j=1}^{n} S_{ij} x_j \frac{\partial}{\partial x_i} + \ldots \ , \tag{62}$$

from where it follows that the n^2 infinitesimal operators are

$$\mathcal{G}_i^j \equiv x_i \frac{\partial}{\partial x_j} \ , \quad i, j = 1, 2, \ldots, n. \tag{63}$$

Operators $\mathcal{G}_i^j$ satisfy the following commutation relations

$$[\mathcal{G}_i^j, \mathcal{G}_k^l] = \mathcal{G}_i^l \delta_{jk} - \mathcal{G}_k^j \delta_{il} \ , \quad i, j, k, l = 1, 2, \ldots, n. \tag{64}$$

6.1.2 Irreducible representations

1. First, there exists a trivial irreducible representation when we put 1 into correspondence to each group element.

2. Further, there exists an irreducible representation realized by matrices themselves. This is an n-dimensional unitary representation.

3. It is possible to construct other irreducible representations based on the transformations of different U(n) tensors. Each tensor representation is uniquely characterized by n integer numbers $[\lambda_1, \lambda_2, \ldots, \lambda_n]$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ which are sometimes displayed pictorially in a Young tableau. Their properties will be considered further while studying the shell model configurations.

6.1.3 Casimir invariants

The group **U(n)** has n different Casimir invariants. The Casimir invariants of the first and the second order in infinitesimal operators have a form:

$$\begin{aligned} C_1[\mathrm{U(n)}] &= \sum_{i=1}^{n} \mathcal{G}_i^i \\ C_2[\mathrm{U(n)}] &= \sum_{i,j=1}^{n} \mathcal{G}_i^j \mathcal{G}_j^i \end{aligned} \tag{65}$$

6.2 SO(n)

6.2.1 Infinitesimal operators

The group **SO(n)** contains $n(n-1)/2$ real parameters, and thus there exist $n(n-1)/2$ infinitesimal operators of this group. In general, **SO(n)** infinitesimal operators Λ_{ij} can be constructed from the corresponding infinitesimal operators of **U(n)** as

$$\Lambda_{ij} = \mathcal{G}_i^j - \mathcal{G}_j^i \,, \quad i < j = 1, 2, \ldots, n. \tag{66}$$

and they satisfy the following commutation relations

$$[\Lambda_{ij}, \Lambda_{kl}] = \Lambda_{il}\delta_{jk} + \Lambda_{jk}\delta_{il} + \Lambda_{lj}\delta_{ik} + \Lambda_{ki}\delta_{jl} \,. \tag{67}$$

6.2.2 Irreducible representations

1. First, there exists a trivial irreducible representation when we put 1 into correspondence to each group element.

2. Further, there exists an irreducible representation realized by matrices themselves. This is an n-dimensional unitary representation.

3. It is possible to construct other irreducible representations based on the transformations of different **SO(n)** tensors. Each tensor representation is uniquely characterized by $n(n-1)/2$ integer numbers $[\lambda_1, \lambda_2, \ldots, \lambda_k]$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ $(k = n(n-1)/2)$ which are sometimes displayed in a Young tableau.

6.2.3 Casimir invariants

The group **SO(n)** has $n/2$ or $(n-1)/2$ different Casimir invariants for n even or odd, respectively. There exists no first order Casimir invariant. The Casimir invariants of the second order in infinitesimal operators has a form:

$$C_2[\mathrm{SO(n)}] = \frac{1}{2} \sum_{i,j=1}^{n} \Lambda_{ij}\Lambda_{ji} \ . \tag{68}$$

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