Lecture2. Introduction to Group Theory

1 Basic definitions

A *group* **G** is a set of distinct elements, for which a law of composition (such as addition, multiplication, matrix multiplication, etc.) is well defined, and which satisfies the following criteria:

1. if G_1 and G_2 are the elements of **G**, then their composition $G_3 = G_1 \cdot G_2$ is also an element of **G**
2. the composition law is associative: $(G_1 \cdot G_2) \cdot G_3 = G_1 \cdot (G_2 \cdot G_3)$
3. there exists an identity element E such that $E \cdot G = G \cdot E = G$ for each element G
4. for each element G from **G**, there exists a unique inverse element G^{-1}, such that $G^{-1} \cdot G = G \cdot G^{-1} = E$.

The number of group elements is called the *order* of the group.

A group containing a finite number of elements is called a *finite* group.

A group containing an infinite number of elements is called an *infinite* group.

An infinite group can be discrete or continuous.

If the number of group elements is denumerably infinite, the group is called *discrete*.

If the number of group elements is non-denumerably infinite, the group is called *continuous*.

In general, the product $G_1 \cdot G_2$ does not have to equal $G_2 \cdot G_1$. However, if $G_1 \cdot G_2 = G_2 \cdot G_1$, the group is called *abelian*.

Examples.

1. The single point set is a group of order 1.
2. Two elements 1 and -1 form a group of order 2. The law of composition is multiplication.
3. The set of all real integers form an infinite discrete group with addition as a law of composition.

4. All non-singular $n \times n$ matrices form a group with multiplication as a law of composition.

5. All possible permutations of n identical objects form a discrete group of order $n!$ (the symmetric group).

The groups of particular interest to physicists are the groups of transformations of a physical system. A transformation which leaves the physical system invariant is called a *symmetry transformation* of the system.

Examples.

1. Inversion in space is a group $\mathbf{I}$ consisting of two elements: E (the identity) and I (the inversion operator).

2. All rotations through an angle $2\pi/n$, where n is an integer, around a fixed axis form a discrete group (the point symmetry group $\mathbf{C}_n$).

3. All rotations around a fixed axis through an arbitrary angle form a continuous group (the special rotational group in two dimensions $\mathbf{SO(2)}$).

4. All rotations in a 3-dimensional space around an arbitrary axis through an arbitrary angle form a continuous group (the special rotational group $\mathbf{SO(3)}$).

5. All rotations and translations in a 3-dimensional space form a continuous group (Euclidean group $\mathbf{E}_3$).

The set of elements $\mathbf{H}$ is said to be a *subgroup* of $\mathbf{G}$ if $\mathbf{H}$ is itself a group under the same law of composition as that of $\mathbf{G}$ and if all elements of H are also elements of $\mathbf{G}$.

Examples

1. For any integer n, $\mathbf{C}_n$ is a subgroup of the group $\mathbf{SO(2)}$.

2. $\mathbf{SO(2)}$ is a subgroup of the group $\mathbf{SO(3)}$.

An element B of the group $\mathbf{G}$ is said to be *conjugate* to element A if we can find an element U in $\mathbf{G}$ such that $UAU^{-1} = B$.

The set of elements which are conjugate to one another is called a *class*.

Example

All rotations through the same angle around axes arbitrary oriented in the space form a class of the group $\mathbf{SO(3)}$.

The groups **G** and **H** are *isomorphic* if they are of the same order and there exists a one-to-one correspondence between the elements of these groups: $G_1 \leftrightarrow H_1$, $G_2 \leftrightarrow H_2$, $G_3 \leftrightarrow H_3, \ldots$. This means that the multiplication tables of these two groups are identical.

Example

The group $\{1, \mathrm{i}, -1, -\mathrm{i}\}$ and the group of rotations around the 4-th order axis $\mathbf{C}_4$ with the elements $\{E, C_4, C_4^2, C_4^3\}$ are isomorphic:

$$\{1 \leftrightarrow E,\ \mathrm{i} \leftrightarrow C_4,\ -1 \leftrightarrow C_4^2,\ -\mathrm{i} \leftrightarrow C_4^3\}$$

.

The *direct product* of the groups **H** of the order l $(H_1, H_2, \ldots, H_l)$and **K** of the order m $(K_1, K_2, \ldots, K_m)$ is defined as a group **G** of the order $n = lm$ consisting of the elements obtained by taking the products of each element of **H** with every element of **K**.

Example

The full orthogonal group **O(3)** is a direct product of the group of 3-dimensional rotations and the group of inversion **I**: **O(3)**=**SO(3)** $\times$ **I**.

2 Point symmetry groups

The transformations which preserve the distances between the points and bring the body into coincidence with itself are called *symmetry transformations*. All symmetry transformations form a *symmetry group* of the body. The symmetry groups of finite bodies which leave at least one point of the body fixed are called *point symmetry groups*.

All point symmetry groups consist of three fundamental operations:

- rotations through an angle $2\pi/n$ (n is integer) around a certain axis: C_n;
- reflection in a symmetry plane: σ;
- combined rotation through an angle $2\pi/n$ (n is integer) around a certain axis and reflection in the perpendicular plane: $S_n = C_n\sigma_h$

Remark

One particular important case of a latter transformation is an *inversion*:

$$I \equiv S_2 = C_2\sigma_h = \sigma_h C_2 \ .$$

The main point-symmetry groups are briefly described below.

1. Groups having a single n-fold rotation axis: $\mathbf{C}_n$.

   Such a group consists of n elements: $E, C_n, C_n^2, \ldots, C_n^{n-1}$. It is a cyclic group.

2. Groups having a single n-fold rotation-reflection axis: $\mathbf{S}_{2n}$.

   Such a group consists of $2n$ elements (the notation $2n$ is introduced because the group is defined only for the even order-fold rotation-reflection axis): $E, S_{2n}, S_{2n}^2, \ldots, S_{2n}^{2n-1}$.

   Remark

   One case of particular importance is the group $\mathbf{S}_2$, often denoted as $\mathbf{I}$, which contains two elements: E and I. If $\mathbf{S}_2$ is a symmetry group of the Hamiltonian of a physical system then the parity is conserved.

3. Groups having a single n-fold and a system of 2-fold axes at right angles to it: dihedral group $\mathbf{D}_n$.

   Such a group consists of $2n$ elements: n elements of the group $\mathbf{C}_n$ and n rotations around the C_2 axes.

   Remark

   $\mathbf{D}_2$ is a symmetry group of the rotational Hamiltonian of an even-A nucleus:

$$H_{rot} = \sum_{k=1}^{3} \frac{\hbar^2}{2\mathcal{I}_k} J_k^2 \ , \tag{1}$$

   where $\mathcal{I}_k$ are the nuclear principle moments of inertia and J_k are the projections of the angular momentum operator on the principle axes.

4. Adjunction of the reflections in a horizontal plane to the group $\mathbf{C}_n$ gives rise to the group $\mathbf{C}_{nh}$.

   This group contains $2n$ elements: n elements of the group $\mathbf{C}_n$, a reflection σ_h and $n-1$ rotation-reflections $C_n\sigma_h$, $C_n^2\sigma_h$, ..., $C_n^{n-1}\sigma_h$.

5. Adjunction of the reflections in the n vertical planes to the group $\mathbf{C}_n$ gives rise to the group $\mathbf{C}_{nv}$.

   This group contains $2n$ elements: n elements of the group $\mathbf{C}_n$ and n reflections in the vertical planes.

6. Adjunction of the reflections in a horizontal plane to the group $\mathbf{D}_n$ gives rise to the group $\mathbf{D}_{nh}$. The horizontal plane automatically gives rise to n vertical planes.

   This group contains $4n$ elements: $2n$ elements of the group $\mathbf{D}_n$, n reflections in the vertical planes and n rotation-reflections.

7. The symmetry group of a regular tetrahedron is known as a *tetrahedral group* $\mathbf{T}$.

   The group contains 12 elements: E, rotations C_2 around three 2-fold axes, rotations C_3 and C_3^2 around four 3-fold axes.

8. Adjunction of the symmetry center to the group $\mathbf{T}$ gives rise to a group $\mathbf{T}_h$=$\mathbf{T}\times$ $\mathbf{I}$.

   The group contains 24 elements.

9. The group containing only rotations around symmetry axes of a cube is known as an *octahedral group* $\mathbf{O}$.

   The group contains 24 elements: E, rotations C_2 around six 2-fold axes, rotations C_3 and C_3^2 around four 3-fold axes, rotations C_4, C_4^2 and C_4^3 around three 4-fold axes.

10. Adjunction of the symmetry center to the group $\mathbf{O}$ gives rise to a group $\mathbf{O}_h = \mathbf{O} \times \mathbf{I}$.

    The group contains 48 elements. This is the full symmetry group of a cube.

11. The symmetry group of a regular icosahedron or regular dodecahedron is known as an *icosahedral group* $\mathbf{Y}$.

    The group contains 60 elements: E, rotations around fifteen 2-fold axes, around ten 3-fold axes and around six 5-fold axes.

12. Adjunction of the symmetry center to the group $\mathbf{Y}$ gives rise to a group $\mathbf{Y}_h = \mathbf{Y} \times \mathbf{I}$.

    The group contains 120 elements.

Remark

The groups $\mathbf{Y}$ and $\mathbf{Y}_h$ cannot be crystalographic symmetry groups, however, they are realized as symmetry groups of molecules, e.g. $H_{12}B_{12}$, or atomic clusters, e.g. fullerene C_{60}.

3 Symmetric group

All permutations of n identical objects

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ p_1 & p_2 & p_3 & \dots & p_n \end{pmatrix} \tag{2}$$

form a group called a *symmetric group* of degree n, denoted as $\mathbf{S}_n$.

The group contains $n!$ elements.

Example

Group $\mathbf{S}_3$ contains 6 elements:

$$\begin{array}{lll} E = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, & \pi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, & \pi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \\ P_{12} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, & P_{23} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, & P_{13} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}. \end{array} \tag{3}$$

Remark

The symmetric group $\mathbf{S}_n$ is of primary importance in quantum mechanics. For any system of n identical particles, the group $\mathbf{S}_n$ is a symmetry group of the Hamiltonian. Thus, the classification of atomic and nuclear states depends essentially on the properties of the group $\mathbf{S}_n$.

4 Continuous matrix groups

In this section we shall consider the continuous groups whose elements can be labelled by a finite set of continuously varying parameters.

Examples

1. The rotation group in two dimensions $\mathbf{SO(2)}$ is characterized by one parameter, the angle of rotation ϕ $(0 \leq \phi < 2\pi)$.
2. All linear transformations of the type

$$x' = ax + b \tag{4}$$

where $-\infty < a, b < +\infty$, form a continuous two-parameter group.

The continuous group is called *compact* if its parameters are restricted in a certain range (e.g., in example 1 above, the angle ϕ of $\mathbf{SO(2)}$ group takes values from a limited domain $0 \leq \phi < 2\pi$).

The continuous group is called *non-compact* if the range of variation of parameters is not specified (e.g., the parameters a and b from (4) can be varied without restrictions between $-\infty$ and $+\infty$).

Remark

If a symmetry group[1] of the Hamiltonian is a compact group, then its spectrum is discrete and of finite dimensions, that correspond to a *bound* spectrum.

The description of a *continuous* spectrum (e.g. scattering states) requires that a symmetry group of the Hamiltonian be a non-compact group.[2]

4.1 Matrix properties

The inverse, transpose, complex conjugate and Hermitian conjugate of a matrix A are denoted by A^{-1}, A^t, A^*, $A^\dagger \equiv (A^t)^*$, respectively.

Matrix relation	Name of matrices
$A = A^t$	Symmetric
$A = -A^t$	Skew symmetric
$A^tA = E$	Orthogonal
$A = A^*$	Real
$A = -A^*$	Imaginary
$A = A^\dagger$	Hermitian
$A = -A^\dagger$	Skew Hermitian
$AA^\dagger = E$	Unitary

The matrix A is called *regular* if its determinant is non-zero.

Some of continuous matrix groups frequently used in physics are listed below.

[1]It should be a dynamical symmetry group, as will be introduced in the forthcoming lectures.

[2]It is important to note that the Coulomb problem is peculiar in having an infinite number of bound states, so that its dynamical group is non-compact.

4.2 General linear group

4.2.1 GL(2)

The linear group in two dimensions $\mathbf{GL}(2)$ is a group of all linear transformations of two coordinates (x, y),

$$\begin{aligned} x' &= a_{11}x + a_{12}y \\ y' &= a_{21}x + a_{22}y \end{aligned} \tag{5}$$

where the parameters a_{11}, a_{12}, a_{21} and a_{22}, as well as the coordinates x and y can be complex and for which the determinant

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0 \ . \tag{6}$$

It is easy to check that all four group criteria are satisfied.

The transformation (5) can be re-written in a matrix form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{7}$$

Thus, we can give an equivalent definition: $\mathbf{GL}(2)$ is a group formed by all regular complex (2×2) matrices.

The group is characterized by eight real parameters (or four complex parameters a_{11}, a_{12}, a_{21}, a_{22}).

4.2.2 GL(n)

All regular complex $(n \times n)$ matrices form the *general linear group* $\mathbf{GL}(\mathrm{n})$, which is characterized by $2n^2$ real parameters.

The group $\mathbf{GL(n)}$ is a non-compact group.

4.3 Unitary groups

4.3.1 U(2)

Let us require that the linear transformations in two dimensions

$$\begin{aligned} x' &= a_{11}x + a_{12}y \\ y' &= a_{21}x + a_{22}y \end{aligned} \tag{8}$$

with

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0 \tag{9}$$

satisfy the additional condition:

$$|x'|^2 + |y'|^2 = |x|^2 + |y|^2. \tag{10}$$

From (10) we can get that the parameters a_{ij} should obey the following relations:

$$\begin{aligned} &|a_{11}|^2 + |a_{21}|^2 = 1 \ , \\ &|a_{12}|^2 + |a_{22}|^2 = 1 \ , \\ &a_{11}a_{12}^* + a_{21}a_{22}^* = 0 \ . \end{aligned} \tag{11}$$

All such transformations form a *unitary group* in two dimensions **U(2)**.

An equivalent definition: **U(2)** is a group formed by all regular unitary (2×2) matrices.

The group is characterized by four real parameters.

4.3.2 U(n)

All unitary $(n \times n)$ matrices form the n^2-parameter *unitary group* **U(n)**.

The group **U(n)** is a subgroup of the group **GL(n)**.

The group **U(n)** is a compact group since $|a_{ij}|^2 \leq 1$.

4.3.3 SU(n)

All unitary $(n \times n)$ matrices whose determinants are equal to $+1$ form the (n^2-1)-parameter **special unitary group SU(n)**.

The group **SU(n)** is a subgroup of the group **U(n)**.

Remark

Besides the rotation group in three dimensions, the unitary groups are among the most frequently used groups in nuclear physics. Some examples are given below.

1. From charge-independence of nuclear forces it follows that the nuclear Hamiltonian is invariant under the transformations of the **SU(2)** group in a charge space (the isospin symmetry).

2. If the SU(3)-symmetry is imposed on the effective two-body shell-model interaction, then the nuclear spectrum will have a rotational structure.

3. From the assumption that nuclear forces are invariant under rotations in spin as well as isospin spaces it follows that the nuclear Hamiltonian has SU(4) symmetry and its energy levels form SU(4)-supermultiplets (Wigner's spin-isospin symmetry).

4.4 Orthogonal groups[3]

4.4.1 O(2)

Let us consider the linear transformations in two dimensions which preserve the distance between two points, i.e.

$$\begin{aligned} x' &= a_{11}x + a_{12}y \\ y' &= a_{21}x + a_{22}y \end{aligned} \tag{12}$$

where a_{11}, a_{12}, a_{21} and a_{22}, as well as x and y take only real values,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0 \tag{13}$$

and

$$x'^2 + y'^2 = x^2 + y^2. \tag{14}$$

[3]Only real orthogonal groups are mentioned here

From(14) we can get that the parameters of such transformations should satisfy the following three relations:

$$\begin{aligned} &a_{11}^2 + a_{21}^2 = 1 \ , \\ &a_{12}^2 + a_{22}^2 = 1 \ , \\ &a_{11}a_{12} + a_{21}a_{22} = 0 \ . \end{aligned} \tag{15}$$

All such transformations form an orthogonal group in two dimensions **O(2)**.

An equivalent definition: **O(2)** is a group formed by all real orthogonal (2×2) matrices.

4.4.2 O(n)

All real orthogonal $(n \times n)$ matrices form the $n(n-1)/2$-parameter *real orthogonal group* **O(n)**.

The group **O(n)** is a subgroup of the group **GL(n)**.

4.4.3 SO(n)

All real orthogonal $(n \times n)$ matrices whose determinants are equal to $+1$ form the *special orthogonal group* **SO(n)**.

The group **SO(n)** is a subgroup of the group **O(n)**.

Examples

1. The rotation group in two dimensions **SO(2)** has one parameter. It can be represented by the matrices

$$\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} , \tag{16}$$

where $0 \le \phi < 2\pi$.

This means that under rotation around the axis perpendicular to the xy-plane, the coordinates (x, y) are transformed as

$$\begin{aligned} x' &= \cos\phi x + \sin\phi y \ , \\ y' &= -\sin\phi x + \cos\phi y \end{aligned} \tag{17}$$

with $0 \le \phi < 2\pi$.

2. The rotation group in three dimensions **SO(3)** is a three parameter group. The most general rotation can be uniquely defined by three parameters, e.g. by three Euler angles $(\alpha,\ \beta,\ \gamma)$:

$$\begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\sin\alpha\cos\gamma & \sin\beta\sin\gamma & \cos\beta \end{pmatrix} . \tag{18}$$

4.4.4 SO(1,1)

Let us consider the linear transformations in two dimensions

$$\begin{aligned} x' &= a_{11}x + a_{12}y \\ y' &= a_{21}x + a_{22}y \end{aligned} \tag{19}$$

where a_{11}, a_{12}, a_{21} and a_{22}, as well as x and y take only real values,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 1 \tag{20}$$

and which preserve the relation:

$$x'^2 - y'^2 = x^2 - y^2. \tag{21}$$

All such transformations form a group in two dimensions **SO(1,1)**.

<u>**Example**</u>

The transformations of the **SO(1,1)** group can be written in the form

$$\begin{aligned} x' &= \gamma x - \gamma\beta(ct) \ , \\ ct' &= -\gamma\beta x + \gamma(ct) \end{aligned} \tag{22}$$

with $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. The invariant form is now $x^2 - c^2t^2$.
This is the 1+1 dimensional Lorentz group.

4.4.5 SO(p,q)

All real $((p+q) \times (p+q))$ matrices whose determinants are equal to $+1$ and which keep invariant the quadratic form

$$x_1^2 + x_2^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2 = inv \tag{23}$$

comprise a group **SO(p,q)**.

<u>**Example**</u>

For $p = 3$ and $q = 1$, the group **SO(3,1)** is known as an extended Lorentz group. The elements of this group keep invariant the following quadratic form:

$$x^2 + y^2 + z^2 - c^2t^2 = inv \ . \tag{24}$$

4.5 Symplectic groups

Let us consider the linear transformations of two points in a plane (x_1, y_1) and (x_2, y_2):

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}y_1 \\ y_1' &= a_{21}x_1 + a_{22}y_1 \end{aligned} \tag{25}$$

and

$$\begin{aligned} x'_2 &= a_{11}x_2 + a_{12}y_2 \\ y'_2 &= a_{21}x_2 + a_{22}y_2 \end{aligned} \tag{26}$$

and let us require that the following relation holds:

$$x'_1 y'_2 - y'_1 x'_2 = x_1 y_2 - y_1 x_2 \ . \tag{27}$$

All such transformations form a group in two dimensions called the *symplectic group*. If the parameters a_{11}, a_{12}, a_{21} and a_{22} are complex then the group is denoted as **Sp(4,C)**. If the parameters a_{11}, a_{12}, a_{21} and a_{22} are real then the group is denoted as **Sp(4,R)**. If we require that the transformations of **Sp(4,C)** be unitary, then we will get the unitary symplectic group denoted as **Sp(4)**.

This can be generalized for n dimensions. Then the corresponding groups will be **Sp(2n,C)**, **Sp(2n,R)** and **Sp(2n)**.

Remark

Symplectic groups often arise in nuclear physics. Some examples are given below.

1. Classification of the many-particle nuclear states in jj-coupling scheme requires the introduction of **Sp(2j+1)** group.

2. Taking into account the particle-hole excitations in the interacting boson model leads to the Sp(2n) symmetries of the Hamiltonian.

References

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