Application of Group Theory to Quantum Mechanics.

Part I
Solutions to Problems

1. *Show that if under transformation of the coordinates the wave functions transforms as (13), then the operators will transform as (14).*

   See Ref. [4].

2. *A system is invariant with respect to the group* $\mathbf{D}_4$ *and its eigenstates can be classified according to the irreducible representations of this group. What will be the degeneracy of the states?*

   $\mathbf{D}_4$ has five irreducible representations: four 1-dimensional and one 2-dimensional, therefore, there will be non-degenerate and 2-fold degenerate levels in the system.

3. *Consider a system having the symmetry* $\mathbf{O}$*. Suppose a perturbation is applied which reduces the symmetry to* $\mathbf{D}_4$*. How will the 2-fold and 3-fold degenerate levels will be splitted?*

   $\mathbf{O}$ has one 2-dimensional irreducible representation ($\tilde{E}$) and two 3-dimensional ones ($\tilde{F}_1$ and $\tilde{F}_2$). The can be decomposed into the irreducible representations of $\mathbf{D}_4$ in the following way (the representations of $\mathbf{D}_4$ are in the right-hand side of the equations):

$$\begin{aligned} \tilde{E} &= A_1 \oplus B_1 \\ \tilde{F}_1 &= A_2 \oplus E \\ \tilde{F}_2 &= B_2 \oplus E \end{aligned}$$

4. *Consider an atom of* ^4He *placed in a crystal of tetrahedral symmetry* $\mathbf{T}_d$ *symmetry. Classify two-electron wave functions.*

   In a strong crystalline field, the levels will be both of pure symmetries A_1, A_2, E, F_1, F_2, and in addition some of them can be grouped into clusters:

$$\begin{aligned} e \times e: &\quad A_1 \oplus A_2 \oplus E; \\ e \times f_1: &\quad F_1 \oplus F_2; \\ f_1 \times f_1: &\quad A_1 \oplus E \oplus F_1 \oplus F_2; \\ f_1 \times f_2: &\quad A_2 \oplus E \oplus F_1 \oplus F_2. \end{aligned}$$